THE POLYNOMIAL-EXPONENTIAL EQUATION $1 + 2^a + 6^b = y^q$

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Abstract. We solve the equation of the title for $q = 3$ and, partially, for $q = 2$. These are the only prime values of q for which there exist integer solutions. Our arguments are based upon off-diagonal Padé approximation to the binomial function.

1. Introduction

Polynomial-exponential equation arise naturally (and, at times, somewhat unnaturally) in a wide variety of mathematical settings. The rather curious equation of the title has been considered by a variety of authors (see e.g [3], [6], [7], [12]) as an example of perhaps the simplest class of polynomials-exponential Diophantine equations whose solutions are in some sense classifiable via the Subspace Theorem of Wolfgang Schmidt, but not, apparently, by simpler means. Recent work [4], [8] solving equations of the shape

$$1 + A^a + B^b = y^q \tag{1}$$

via local methods, appears unable to treat the equation of the title (indeed, the techniques of [4] and [8] both require that $A \not\equiv B \pmod 2$). As far as we are aware, the title equation is the only case of (1) with $\max\{A, B\} \leq 6$ that remains unsolved. Regrettably, this paper is unable to completely rectify this fault.

Strong partial results are already available in the literature. Indeed, a very special case of a theorem of Corvaja and Zannier [5] (based upon the aforementioned Subspace Theorem) implies that, for fixed exponent q, the title equation has at most finitely many solutions. Further, work of the author with Bugeaud and Mignotte (Theorem 3 of [3]) implies that the equation of the title has no solutions unless we have $q \in \{2, 3, 6\}$. This latter result depends fundamentally upon bounds for linear forms in 2-adic and complex logarithms. In the paper at hand, we will appeal to explicit off-diagonal Padé approximations to the function $(1 + z)^{1/q}$ to (partially) treat these remaining cases. We prove the following.

Theorem 1. *The equation*

$$1 + 2^a + 6^b = y^2 \tag{2}$$

has only the solutions $(a, b, y) = (1, 1, 3)$ *and* $(3, 3, 15)$ *in nonnegative integers with* $a \leq b$. *The equation*

$$1 + 2^a + 6^b = y^3 \tag{3}$$

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has only the solutions $(a,b,y)=(0,1,2)$ *and* $(9,3,9)$ *in nonnegative integers.*

This resolves a problem of Luca [9] (corresponding to the special case of equation (2) with $a=b$). We note that the only other known solutions to equation (2) (in the remaining case $a>b$) are with $(a,b,y)=(1,0,2)$ and $(9,3,27)$. We strongly suspect that they are the only ones.

We will actually prove something somewhat stronger than Theorem 1 (in the case of equation (2)).

Proposition 1.1. *If there exist nonnegative integers* a,b *and* y *satisfying (2), then either* $(a,b,y)=(1,0,2),(1,1,3)$ *or* $(3,3,15)$*, or* a *and* b *are positive and we have*

$$a/b\in(1.349,4.250)\cup(6.166,9.943).$$

Again, it is likely that this last condition can be replaced by $(a,b,y)=(9,3,27)$.

2. The square case : Proposition 1.1

In this section, we will restrict our attention to (2). If $b=0$ in this equation, it is straightforward to show that $a=1$, corresponding to $(a,b,y)=(1,0,2)$. Similarly, there are no solutions with $a=0$. If $b=1$, then we have a solution $(a,b,y)=(1,1,3)$ and, modulo 4, no others. For $b\geq 2$, we find that, modulo 9, $a\equiv 3 \pmod 6$, whereby, modulo 7, b is also odd. We will therefore suppose, for the remainder of this section, that a and b are odd integers (with $3\mid a$) and $\min\{a,b\}\geq 3$.

Our starting point will be some "off-the-shelf" results from explicit Diophantine approximation (though we will have need of more specialized ones later). From Corollary 1.7 of [1], we have, for a and b odd, that

$$\left|y^2-2^a\right|>2^{0.26a}\quad\text{and}\quad\left|y^2-6^b\right|>6^{0.27b},\tag{4}$$

unless, in the first case, $(y,a)=(\pm 3,3)$ or $(\pm 181,15)$. A short calculation thus allows us to conclude that either $(a,b)=(3,3)$ or $(9,3)$, or that $\min\{a,b\}>10$, whereby, from (4),

$$b<\frac{2}{0.54}\frac{\log\left(1+2^a\right)}{\log 6}<1.433\,a\quad\text{and}\quad a<\frac{2}{0.52}\frac{\log\left(1+6^b\right)}{\log 2}<9.943\,b.\tag{5}$$

We therefore have that

$$a/b\in(0.697,9.943)\,.\tag{6}$$

To sharpen this conclusion, we will begin by supposing that $a\leq b$, whereby we may write $y=(-1)^\delta+k\cdot 2^{a-1}$, for k an integer and $\delta\in\{0,1\}$. It follows that

$$(-1)^\delta k+k^2\cdot 2^{a-2}=1+2^{b-a}3^b.$$

Since $b<1.433a$ and $a>10$, we have that $b-a<a-2$ and so

$$3^b-k^2\cdot 2^{2a-b-2}=3^b-2\left(k\cdot 2^{a-\frac{b+3}{2}}\right)^2=\frac{(-1)^\delta k-1}{2^{b-a}}$$

is an integer. We note that

$$k\leq\frac{y+1}{2^{a-1}}<\frac{6^{b/2}+2}{2^{a-1}},$$

whence

$$\left|3^b-2\left(k\cdot 2^{a-\frac{b+3}{2}}\right)^2\right|<\frac{6^{b/2}+3}{2^{b-1}}<3\cdot(3/2)^{b/2}.\tag{7}$$

If, on the other hand, we have $a > b$, then we may write

$$y = (-1)^\delta + k \cdot 2^{b-1},$$

for $k \in \mathbb{N}$ and $\delta \in \{0, 1\}$, and so

$$(-1)^\delta k + k^2 \cdot 2^{b-2} = 2^{a-b} + 3^b \tag{8}$$

and

$$3^b - 2\left(k \cdot 2^{\frac{b-3}{2}}\right)^2 = (-1)^\delta k - 2^{a-b}. \tag{9}$$

If we suppose further that $2^a \leq 6^{b/2}$, then $y < 1 + 6^{b/2}$ and so

$$k \leq \frac{y+1}{2^{b-1}} < \frac{2 + 6^{b/2}}{2^{b-1}},$$

whereby

$$\left| 3^b - 2\left(k \cdot 2^{\frac{b-3}{2}}\right)^2 \right| < \frac{2 + 6^{b/2}}{2^{b-1}} + (3/2)^{b/2} < 4 \cdot (3/2)^{b/2}. \tag{10}$$

Combining (7) and (10), it follows that, in all cases where $2^a \leq 6^{b/2}$, we have

$$\left| 3^b - 2T^2 \right| < 4 \cdot (3/2)^{b/2}, \tag{11}$$

for T an integer (and b odd). If, however, $6^{b/2} < 2^a < 6^b$, we have that $y \leq \sqrt{2}\, 6^{b/2}$ and so, from (9) and the fact that $b > 10$,

$$\left| 3^b - 2\left(k \cdot 2^{\frac{b-3}{2}}\right)^2 \right| < 3 \cdot (3/2)^{b/2} + 2^{a-b}. \tag{12}$$

Our immediate goal will be to show that inequality (11) is never satisfied, at least provided $b \geq 8$, and that inequality (12) cannot hold for "small" values of a. We note that, from a result of Ridout [11], given $\epsilon > 0$, we have, writing $b = 2b_0 + 1$, that

$$\left| \sqrt{3/2} - \frac{T}{3^{b_0}} \right| \gg_\epsilon 3^{-(1+\epsilon)b_0}$$

and hence there exists a constant $c(\epsilon) > 0$ such that

$$\left| 3^b - 2T^2 \right| > c(\epsilon)\, 3^{(1-\epsilon)b/2}, \tag{13}$$

for b odd and $T \in \mathbb{Z}$. In particular, there are at most finitely many solutions to (11). For our purposes, however, we require an effective, explicit lower bound (of necessity, somewhat weaker than (13)). We prove the following.

Proposition 2.1. *If b and T are nonnegative integers, with $b \geq 7$ odd, then*

$$\left| 3^b - 2T^2 \right| > 1.274^b.$$

Proof. To derive this result, let us begin by defining, for n_1 and n_2 nonnegative integers, polynomials in $\mathbb{Q}[z]$

$$P_{n_1,n_2}(z) = \sum_{k=0}^{n_1} \binom{n_2 + 1/2}{k} \binom{n_1 + n_2 - k}{n_2} z^k \tag{14}$$

and

$$Q_{n_1,n_2}(z) = \sum_{k=0}^{n_2} \binom{n_1 - 1/2}{k} \binom{n_1 + n_2 - k}{n_1} z^k. \tag{15}$$

These are, up to scaling, the $[n_1, n_2]$-Padé approximants to $(1+z)^{1/2}$ and satisfy the relation

$$P_{n_1,n_2}(z) - (1+z)^{1/2}\ Q_{n_1,n_2}(z) = z^{n_1+n_2+1}\, E_{n_1,n_2}(z), \tag{16}$$

where

$$E_{n_1,n_2}(z) = \frac{(-1)^{n_2}\,\Gamma(n_2+(q+1)/q)}{\Gamma(-n_1+1/q)\Gamma(n_1+n_2+2)} F(n_1+(q-1)/q, n_2+1, n_1+n_2+2, -z),$$

and F is the hypergeometric function

$$F(a,b,c,-z) = 1 - \frac{a\cdot b}{1\cdot c}z + \frac{a\cdot(a+1)\cdot b\cdot(b+1)}{1\cdot 2\cdot c\cdot(c+1)}z^2 - \cdots .$$

For our purposes, it is enough to note the following, combining Lemmata 3.1 and 4.1 of [1] :

Lemma 2.2. *Suppose that z is a real number with $|z| \leq 1/2$ and that n_1 and n_2 are positive integers and $\alpha \geq 3/2$ a real number satisfying*

$$0 \leq \alpha n_1 - n_2 < 2(\alpha - 1). \tag{17}$$

Define

$$r(\alpha,z) = -\frac{1}{2z}\left((\alpha+1)+(\alpha-1)z - \sqrt{((\alpha+1)+(\alpha-1)z)^2+4z}\right)$$

and

$$F_{\alpha,z} = \frac{(1+zr(\alpha,z))^{\alpha}}{r(\alpha,z)\,(1-r(\alpha,z))^{\alpha}}.$$

Then we have

$$|P_{n_1,n_2}(z)| < 2(\alpha+1)\ (F_{\alpha,z})^{n_1} \tag{18}$$

and

$$\left|P_{n_1,n_2}(z) - (1+z)^{1/2}\ Q_{n_1,n_2}(z)\right| < (\alpha+1)^2\,|z|^{3-2\alpha}\ \left(|z|^{-(\alpha+1)}\,F_{\alpha,z}\right)^{-n_1}. \tag{19}$$

Continuing with the proof of Proposition 2.1, let us write, given a nonzero integer s, $\nu_2(s)$ for the largest power of 2 dividing s. If s and t are nonzero integers, define $\nu_2(s/t) = \nu_2(s) - \nu_2(t)$. Then we have, given an integer n and a nonnegative integer k,

$$\nu_2\left(\binom{n\pm 1/2}{k}\right) = -k - \sum_{j=1}^{\infty}\left[\frac{k}{2^j}\right], \tag{20}$$

so that, in particular, $2^{2k}\binom{n\pm1/2}{k}$ is an integer. If we substitute $z = -1/243$ into (16) (taking advantage of the identity $3^5 - 1 = 2\cdot 11^2$) and suppose that $n_2 > n_1$, then, multiplying by $2^{2n_2}\cdot 3^{5n_2+2}$, the left hand side of (16) becomes

$$\Omega = 2^{2(n_2-n_1)}3^{5(n_2-n_1)+2}\mathcal{P}_{n_1,n_2} - \sqrt{\frac{2}{3}}\cdot 11\,\mathcal{Q}_{n_1.n_2}, \tag{21}$$

where

$$\mathcal{P}_{n_1,n_2} = 2^{2n_1}\,3^{5n_1}\,P_{n_1,n_2}\left(\frac{-1}{243}\right)$$

and

$$\mathcal{Q}_{n_1.n_2} = 2^{2n_2}\,3^{5n_2}\,Q_{n_1,n_2}\left(\frac{-1}{243}\right)$$

are integers. Let us, for future use, define $G_{n_1,n_2} = \gcd(P_{n_1,n_2}, Q_{n_1,n_2})$. It is important for us (and somewhat nontrivial) that G_{n_1,n_2} grows exponentially in the parameters n_1 and n_2. Indeed, G_{n_1,n_2} is divisible by all primes in certain intervals with lengths exceeding a constant multiple of $\min\{n_1, n_2\}$ (see e.g. the proof of Proposition 5.2 of [1]). By the Prime Number Theorem, this quantity is therefore at least exponentially large in $\min\{n_1, n_2\}$.

Next, suppose that b is an odd, positive integer, say $b = 2b_0 + 1$, and that T is an integer. Set

$$\Upsilon = \sqrt{2/3}\, T - 3^{b_0}.$$

We will begin by assuming that $b \geq 307000$. Choose positive integers n_1 and n_2 such that

$$n_1 = \left[\frac{b}{55}\right] + 1 \;\text{ and }\; n_2 = \left[\frac{13n_1}{2}\right],$$

where by $[x]$ we mean the greatest integer not exceeding a real number x (so that, in particular, $n_1 \geq 5582$). Then

$$5(n_2 - n_1) + 2 \geq \frac{55n_1 - 1}{2} \geq \frac{b}{2}$$

and hence $5(n_2 - n_1) + 2 \geq b_0$. We may also readily observe that n_1 and n_2 satisfy (17) with $\alpha = 6.5$. We thus have

$$11\, Q_{n_1,n_2} \Upsilon + T\Omega = 3^{b_0} \left(2^{2(n_2-n_1)} 3^{5(n_2-n_1)+2-b_0} P_{n_1,n_2} T - 11\, Q_{n_1,n_2} \right). \tag{22}$$

The right hand side of this is an integer multiple of $3^{b_0} G_{n_1,n_2}$ and is, in fact, nonzero. To see this last point, observe that, from (20),

$$\nu_2\left(2^{2n_2}\binom{n_1 - 1/2}{k}\right) = 2n_2 - k - \sum_{j=1}^{\infty}\left[\frac{k}{2^j}\right],$$

for each $0 \leq k \leq n_2$, and, in particular, that

$$\nu_2\left(2^{2n_2}\binom{n_1 - 1/2}{k}\right) \geq n_2 + 1 - \sum_{j=1}^{\infty}\left[\frac{n_2}{2^j}\right],$$

for each k with $0 \leq k \leq n_2 - 1$, while

$$\nu_2\left(2^{2n_2}\binom{n_1 - 1/2}{n_2}\right) = n_2 - \sum_{j=1}^{\infty}\left[\frac{n_2}{2^j}\right].$$

It follows from (15) that

$$\nu_2(Q_{n_1,n_2}) = n_2 - \sum_{j=1}^{\infty}\left[\frac{n_2}{2^j}\right] \leq \frac{\log n_2}{\log 2}$$

and hence, since this is strictly smaller than $2(n_2 - n_1)$,

$$\nu_2\left(11\, Q_{n_1,n_2} \Upsilon + T\Omega\right) = \nu_2(Q_{n_1,n_2}),$$

whereby $11\, Q_{n_1,n_2} \Upsilon + T\Omega \neq 0$. We thus have

$$|11 Q_{n_1,n_2} \Upsilon + T\Omega| \geq 3^{b_0}\, G_{n_1,n_2}. \tag{23}$$

On the other hand, if we suppose that

$$\left|3^b - 2T^2\right| \leq 3^{b/2},$$

say, then, $3^{b_0} < T < 2 \cdot 3^{b_0}$ and so, from Lemma 2.2,

$$|T\,\Omega| < 2^{2n_2-1} \cdot 5^2 \cdot 3^{b_0+5n_2+54-37.5n_1}\, F_{6.5,-1/243}^{-n_1},$$

whence, since $n_2 \leq \frac{13n_1+1}{2}$,

$$|T\,\Omega| < 2^{13n_1} \cdot 5^2 \cdot 3^{b_0+56.5-5n_1}\, F_{6.5,-1/243}^{-n_1}.$$

From $F_{6.5,-1/243} = 18.943966\ldots$, it follows that

$$|T\,\Omega| < 5^2 \cdot 3^{b_0+56.5}\, 1.779561^{n_1}. \tag{24}$$

On the other hand, appealing to Proposition 5.2 of [1], we have that

$$G_{n_1,n_2} \geq e^{-19.408}\, 1.807^{n_1}$$

and so, since $n_1 \geq 5582$,

$$|T\,\Omega| < \frac{1}{2}\, 3^{b_0}\, G_{n_1,n_2}. \tag{25}$$

We thus have, from (23), that

$$|Q_{n_1,n_2}\Upsilon| \geq \frac{1}{22}\, 3^{b_0}\, G_{n_1,n_2} \geq \frac{1}{22}\, e^{-19.408}\, 1.807^{n_1}\, 3^{b_0}. \tag{26}$$

From (24) and the fact that $T > 3^{b_0}$, we have

$$|\Omega| < 5^2 \cdot 3^{56.5}\, 1.779561^{n_1}$$

and so, since

$$\sqrt{\frac{2}{3}} \cdot 11\ \ |Q_{n_1.n_2}| \leq 2^{2(n_2-n_1)} 3^{5(n_2-n_1)+2}\, |P_{n_1,n_2}| + |\Omega|\,,$$

applying Lemma 2.2,

$$|Q_{n_1.n_2}| \leq e^{63.1} \cdot 1.779561^{n_1} + 469 \cdot e^{47.6573n_1} < 470 \cdot e^{47.6573n_1}.$$

With (26), this implies that

$$|\Upsilon| > e^{-28.7}\, e^{-47.066n_1}\, 3^{b_0}.$$

Since $n_1 < \frac{b}{55} + 1$, we thus have

$$|\Upsilon| > e^{-76.32}\, 1.35858^{-b}.$$

Now we write

$$\left|3^b - 2T^2\right| = 3|\Upsilon| \left(\sqrt{2/3}\, T + 3^{b_0}\right),$$

so that

$$\left|3^b - 2T^2\right| > \left(\sqrt{3} + \sqrt{2}\right) 3^{b/2}\, e^{-76.32}\, 1.35858^{-b},$$

i.e.

$$\left|3^b - 2T^2\right| > e^{-75.18}\, 1.2748979^b > 1.274^b,$$

where the last inequality is a consequence of the fact that $b \geq 307000$.

To check the desired inequality for odd values of b with $7 \leq b < 307000$ is a routine matter; one can simply verify it by brute force for small values of b and otherwise search for long strings of zeros or twos in the ternary expansion of $\sqrt{3/2}$ (the reader is directed to [1], in particular to Lemma 9.1 of [1] and the remarks following it, for details of such a computation). The fact that we find none completes the proof of Proposition 2.1. □

Applying Proposition 2.1 to (10), it follows that $b \le 35$; we find no new solutions to (2) with $10 < b \le 35$ and $2^a \le 6^{b/2}$. Assuming next that $6^{b/2} < 2^a < 6^b$, if we suppose further that $a \le 1.349b$, then, from Proposition 2.1 and inequality (7),

$$3 \cdot (3/2)^{b/2} + 2^{0.349b} > 1.274^b$$

and so $b \le 94$. Again, a short check confirms the absence of new solutions to (2) and hence that $a/b > 1.349$. With (6), we thus have

$$a/b \in (1.349, 9.943)\,. \tag{27}$$

To complete the proof of Proposition 1.1, it remains to show that there are no solutions to equation (2) with $a/b \in [4.250, 6.166]$ and $\min\{a,b\} > 10$. Suppose that we have such a solution. Then $2^a > 1000 \cdot 6^b$, whereby $k < 1.1 \cdot 2^{\frac{a-2b}{2}}$ and we have, from (8),

$$\left|2^{a-2b+2} - k^2\right| = \frac{\left|(-1)^\delta k - 3^b\right|}{2^{b-2}} < 9 \cdot \max\{2^{\frac{a}{2}-2b}, (3/2)^b\}.$$

If $2^{\frac{a}{2}-2b} > (3/2)^b$, then (4) implies that

$$9 \cdot 2^{\frac{a}{2}-2b} > 2^{0.26(a-2b+2)},$$

whence

$$2\log 3 + 0.24\, a \log 2 > 0.52 \log 2 + 1.48\, b \log 2,$$

i.e.

$$a > \frac{37}{6}\, b - 11.05.$$

We thus have $a > 6.166b$ for all $b \ge 16575$, a contradiction. Once again, smaller values of b fail to lead to new solutions. If, on the other hand, $2^{\frac{a}{2}-2b} \le (3/2)^b$, then (4) yields

$$9 \cdot (3/2)^b > 2^{0.26(a-2b+2)},$$

whereby

$$2\log 3 + b\left(\log(3/2) + 0.52\, \log 2\right) > 0.52\, \log 2 + 0.26\, a \log 2,$$

and so

$$a < 4.2499b + 1.84 \le 4.25b,$$

where the last inequality holds for $b \ge 18400$. Another short calculation finishes the proof of Proposition 1.1.

3. An ineffective approach to equation (2)

Before we proceed with our treatment of equation (3), we will indicate how ineffective results from Diophantine approximation imply finiteness for equation (2) (as mentioned earlier, this follows immediately from work of Corvaja and Zannier [5]). Such results can be used to bound the *number* of solutions to (2), but not their size. Specifically, we will show how such a conclusion follows from only the one-dimensional version of Schmidt's Subspace Theorem, which, in the strength we require, dates back to Ridout [11]. Taking Proposition 1.1 as our starting point, given a nonzero integer b_0 and $\epsilon > 0$, we have from [11] the inequality

$$\left|\sqrt{6} - \frac{k \cdot 2^{b_0}}{3^{b_0}}\right| \gg_\epsilon 3^{-\left(\frac{\log(3/2)}{\log 3}+\epsilon\right)b_0}$$

and hence, writing $b = 2b_0 + 1$, the existence of a constant $c(\epsilon) > 0$ such that

$$\left|3^b - k^2 \cdot 2^{b-2}\right| > c(\epsilon)\, 3^{\left(1-\frac{\log(3/2)}{2\log 3}-\epsilon\right)b}.$$

From (8), it follows that

$$\left|2^{a-b} - (-1)^\delta k\right| > c(\epsilon)\, 3^{\left(1-\frac{\log(3/2)}{2\log 3}-\epsilon\right)b}.$$

Choosing ϵ suitably small, this implies, with Proposition 1.1, that there are at most finitely many solutions to (2) with $a < 2.29b$. Assuming $a \geq 2b$, then, we may rewrite (8) as

$$3^b - 2^{b-2}\left(2^{a-2b+2} - k^2\right) = (-1)^\delta k.$$

Again appealing to Ridout [11], this time in the form considered by Mahler [10], we have

$$\left|\ 3^b - 2^{b-2}\left(2^{a-2b+2} - k^2\right)\right| \gg_\epsilon 2^{(1-\epsilon)b}.$$

Since $k \ll 2^{\frac{a}{2}-b}$, it follows that there are at most finitely many solutions to (2), with, say, $a < 3.9b$.

Finally, for $3.9b \leq a < 9.943b$, we write, as in the preceding section,

$$\left|2^{a-2b+2} - k^2\right| = \frac{\left|(-1)^\delta k - 3^b\right|}{2^{b-2}} \ll \max\{2^{\frac{a}{2}-2b}, (3/2)^b\}$$

and compare this inequality to the lower bound

$$\left|2^{a-2b+2} - k^2\right| \gg_\epsilon \max 2^{(1-\epsilon)\left(\frac{a}{2}-b\right)}$$

coming from [11]. Taking, say, $\epsilon = 1/4$ implies the desired inequality, with at most finitely many exceptions.

4. The cubic case

To complete the proof of Theorem 1, we must treat equation (3). Let us suppose that there exist nonnegative integers a, b and y satisfying (3). If $b = 0$, then we reach a contradiction, modulo 4. If $b = 1$, then, modulo 7, we necessarily have that $3 \mid a$, say $a = 3a_0$, whereby

$$y^3 - 2^{3a_0} = 7,$$

and so $(a, b, y) = (0, 1, 2)$. We may thus assume that $b \geq 2$ so that, modulo $3^2 \cdot 7$, $3 \mid a$ (whence b is odd and $3 \mid y$).

Suppose first that $a > b$, whereby

$$y = 1 + k \cdot 2^b,$$

for k an odd integer (with, necessarily, $k \equiv 1 \pmod 3$). It follows that

$$2^{a-b} + 3^b = 3k + 3k^2 \cdot 2^b + k^3 \cdot 2^{2b}. \tag{28}$$

If $k = 1$, we thus have

$$2^{a-b} + 3^b = 3 + 3 \cdot 2^b + 2^{2b}$$

and hence, if additionally $a \leq 3b - 1$, then

$$2^{2b-1} + 3^b \geq 3 + 3 \cdot 2^b + 2^{2b}$$

and so

$$3^b \geq 3 + 3 \cdot 2^b + 2^{2b-1},$$

an immediate contradiction. If $a = 3b$,

$$3^{b-1} = 1 + 2^b$$

whence $b \leq 3$ (so that $(a,b) \in \{(6,2),(9,3)\}$). The latter of these corresponds to the solution $(a,b,y) = (9,3,9)$. If $k=1$, we may therefore suppose, since $3 \mid a$, that $a \geq 3b+3$. Otherwise, we have $k \geq 7$ so that, from (28), it is again easy to show that $a \geq 3b+3$. We may thus write $a = 3b+3t$ for a positive integer t, whence (28) becomes

$$2^{2b+3t} + 3^b = 3k + 3k^2 \cdot 2^b + k^3 \cdot 2^{2b},$$

i.e.

$$2^{3t} - k^3 = \frac{3k + 3k^2 \cdot 2^b - 3^b}{2^{2b}}.$$

Since k is odd, the left-hand-side here is a nonzero integer and hence

$$3k^2 - 3k + 1 \leq \left|2^{3t} - k^3\right| \leq \frac{3k + 3k^2 \cdot 2^b + 3^b}{2^{2b}},$$

a contradiction unless $k=1$ and $b=2$ (in which case, $a \geq 8$ contradicts (28)).

It follows that we may assume that $a \leq b$. If $a = b$, then the fact that $3 \mid a$ leads to an immediate contradiction. We thus have $a < b$, so that $y = 1 + k \cdot 2^a$ for an odd positive integer k and we may write

$$1 + 2^{b-a}3^b = 3k + 3k^2 \cdot 2^a + k^3 \cdot 2^{2a}. \tag{29}$$

If $b < 2a$, then it follows that $2^{b-a} \mid 3k-1$, so that, in particular, $3k - 1 \geq 2^{b-a}$ and hence

$$2^{b-a}3^b > \frac{1}{27}\, 2^{3b-a}, \quad \text{i.e. } 3^{b+3} > 2^{2b},$$

whereby $b \leq 11$. Since $a < b < 2a$, a short check reveals no solutions in this case.

It follows that $b \geq 2a$. We will proceed by appealing to Diophantine consequences of explicit lower bounds for rational approximation to $\sqrt[3]{6}$. In particular, we will use Theorem 6.1 of [2] which implies the inequality

$$\left|A^3 - 6B^3\right| > \max\{|A|,|B|\}^{0.65}, \tag{30}$$

valid for all nonzero integers A and B, provided $(A,B) \neq \pm(467,257)$. If $b \equiv 0 \pmod 3$, say $b = 3b_0$, then

$$1 + 2^a = y^3 - 6^b \geq \left(6^{b_0}+1\right)^3 - 6^{3b_0} > 3 \cdot 6^{2b/3},$$

a contradiction. We thus have $b \equiv \pm 1 \pmod 3$, say $b = 3b_0+1$ or $b = 3b_0+2$, for b_0 a nonnegative integer. In the first case, we apply (30) with $A = y$ and $B = 6^{b_0}$ to conclude that

$$1 + 2^a = \left|y^3 - 6^b\right| > 6^{0.65b_0} = 6^{0.65(b-1)/3},$$

while, in the latter case, we take $A = 6^{b_0+1}$ and $B = y$ to find that

$$1 + 2^a = \left|y^3 - 6^b\right| > \frac{1}{6}\, 6^{0.65(b_0+1)} = 6^{0.65(b+1)/3-1}.$$

In either situation, we therefore have that

$$2^{b/2} \geq 2^a > 0.24 \cdot 6^{0.65b/3} - 1,$$

whereby $b \leq 34$. Using that $b \geq 2a$, we check quickly that no additional solutions accrue. This concludes our proof.

It is perhaps worthwhile to observe that while inequality (30) is quite general (and sufficient for our purposes), if we really wish to use the additional arithmetic data that either A or B is a power of 6, then stronger inequalities may be obtained through arguments similar to those given in [1] in the quadratic case. In particular, one may, for example, prove that

$$\left|y^3 - 6^b\right| > 6^{0.35b},$$

provided the quantity on the left-hand-side is nonzero.

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